A classification of semi-equivelar gems on the double torus

Anshu Agarwal, Biplab Basak, and Debolina Ghosh [1]

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India.[2]

Abstract

Semi-equivelar structures arise naturally in the study of highly symmetric decompositions of manifolds. A semi-equivelar gem of a PL d-manifold is a regular properly colored graph representing the manifold that admits a regular embedding on a surface such that the cyclic sequence of face degrees around every vertex is identical. These structures encode both the topology of the manifold and the combinatorics of the embedding, and hence provide a useful framework for studying manifolds using graph-theoretic and combinatorial techniques.

In recent years, semi-equivelar embeddings have attracted considerable attention due to their connections with equivelar maps, Archimedean tilings, and colored graph representations of manifolds. Since gems provide a combinatorial model for PL manifolds, investigating the possible semi-equivelar embeddings of such graphs on surfaces helps reveal structural constraints imposed by both the topology of the surface and the coloring of the graph.

Previous work classified semi-equivelar gems on surfaces with Euler characteristic $\chi \geq -1$. In particular, Basak and Binjola [3] classified such embeddings for surfaces with nonnegative Euler characteristic, and Agarwal and Basak [1] extended this classification to surfaces with Euler characteristic -1. These results provide a complete understanding of semi-equivelar gems on surfaces up to Euler characteristic -1.

A natural next step is to investigate the case of surfaces with Euler characteristic -2, namely the orientable surface of genus two (the double torus). Understanding semi-equivelar embeddings in this setting is important for extending the classification program to surfaces of higher genus.

This naturally leads to the following question.

Question 1. Which semi-equivelar colored graphs admit regular embeddings on the double torus?

Question 2. For each admissible semi-equivelar type on the double torus, does there exist a semi-equivelar gem realizing that type?

Question 3. For which numbers of colors (3-colored, 4-colored, or 5-colored) can a semi-equivelar gem be embedded on the double torus?

Question 4. What is the minimum number of vertices required to realize each semi-equivelar type on the double torus?

[1] Will present the paper

[2] *E-mail addresses:* maz228084@maths.iitd.ac.in (A. Agarwal), biplab@iitd.ac.in (B. Basak), and maz258290@maths.iitd.ac.in (D. Ghosh).

Question 5. Can the classification of semi-equivelar gems be extended to orientable surfaces of higher genus?

Question 6. Which PL d-manifolds can be represented by semi-equivelar gems whose regular embeddings occur on surfaces of genus two?

Question 7. Is it possible to obtain a complete classification of semi-equivelar gems embedded on orientable surfaces of genus $g \geq 3$? More precisely, determine all semi-equivelar types of colored graphs that admit regular embeddings on such surfaces and characterize those that can be realized by gems of PL d-manifolds.

The study of equivelar and semi-equivelar structures originates in the work of McMullen, Schulz, and Wills [11], who introduced equivelar polyhedral manifolds and showed the existence of such manifolds with prescribed combinatorial types. Later, they constructed infinite families of equivelar surfaces with arbitrarily large genus embedded in $\mathbb{E}^3$ [12]. Subsequent research focused on classifying equivelar and semi-equivelar maps on surfaces of small genus. Datta and Nilakantan [8] classified simplicial equivelar polyhedra with at most eleven vertices, while formulas for counting equivelar triangulations and quadrangulations of the torus were obtained in [5].

To generalize equivelar maps, the notion of *semi-equivelar maps* was introduced in [10]. In such maps the cyclic sequence of face types around each vertex is the same, although the faces themselves may have different sizes. This concept led to several classification results, including semi-equivelar maps on the torus and Klein bottle [6], the 2-sphere [7], and surfaces with Euler characteristic -1 [4].

Another important framework in combinatorial topology is that of *graph-encoded manifolds* (gems). A gem of a closed PL d-manifold is a regular $(d+1)$-colored graph encoding the manifold structure. Every closed PL d-manifold admits such a representation, and a given manifold may have several non-isomorphic gems. Moreover, gems admit regular embeddings on surfaces [9]. The classification of surfaces using gems was established in [2]. The concept of *semi-equivelar gems*, combining semi-equivelar embeddings with gems, was introduced in [3].

Main Results

In this paper we extend the classification of semi-equivelar gems to embeddings on the *double torus*, the orientable surface of genus 2.

Theorem 1. *All semi-equivelar colored graphs that admit regular embeddings on the double torus belong to one of the following* 31 *types:*

$$(4^5), (6^4), (4^3, 6), (4^3, 8), (4^3, 12), (4^2, 6^2), (4, 6, 4, 6), (4^2, 8^2), (4, 8, 4, 8), (8^3), (10^3),$$

$$(6^2, 8), (6^2, 10), (6^2, 12), (6^2, 18), (10^2, 4), (12^2, 4), (16^2, 4), (8^2, 6), (12^2, 6),$$

$$(4, 6, 14), (4, 6, 16), (4, 6, 18), (4, 6, 20), (4, 6, 24), (4, 6, 36),$$

$$(4, 8, 10), (4, 8, 12), (4, 8, 16), (4, 8, 24), (4, 10, 20).$$

These types consist of one 5-colored graph, eight 4-colored graphs, and twenty-two 3-colored graphs.

Theorem 2. *For each of the* 31 *admissible types listed above, there exists a semi-equivelar gem whose regular embedding on the double torus realizes that type.*

Thus, we obtain a complete classification of semi-equivelar gems embedded on the double torus, extending the classification program initiated in earlier works.

The paper is available on : `https://arxiv.org/abs/2512.13135`.

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